

ON A CONJECTURE OF KANEKO AND OHNO

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Abstract. Let $X_0^*(k, n, s)$ denote the sum of all multiple zeta-star values of weight k , depth n and height s . Kaneko and Ohno conjecture that for any positive integers m, n, s with $m, n \geq s$, the difference $(-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s)$ can be expressed as a polynomial of zeta values with rational coefficients. We give a proof of this conjecture in this paper.

Keywords: multiple zeta-star value, generalized hypergeometric function

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1. INTRODUCTION

Let $\mathbf{k} = (k_1, \dots, k_n)$ be a sequence of positive integers with $k_1 > 1$, the weight $\text{wt}(\mathbf{k})$, depth $\text{dep}(\mathbf{k})$ and height $\text{ht}(\mathbf{k})$ are defined by

$$\text{wt}(\mathbf{k}) = k_1 + \dots + k_n, \text{dep}(\mathbf{k}) = n, \text{ht}(\mathbf{k}) = \#\{i \mid k_i \geq 2\},$$

respectively. For such a sequence \mathbf{k} , there are two well studied real numbers: multiple zeta value $\zeta(\mathbf{k})$ defined by

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_n) = \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \dots m_n^{k_n}},$$

and multiple zeta-star value $\zeta^*(\mathbf{k})$ defined by

$$\zeta^*(\mathbf{k}) = \zeta^*(k_1, \dots, k_n) = \sum_{m_1 \geq \dots \geq m_n \geq 1} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}.$$

We call the values $\zeta(\mathbf{k})$ and $\zeta^*(\mathbf{k})$ with weight $\text{wt}(\mathbf{k})$, depth $\text{dep}(\mathbf{k})$ and height $\text{ht}(\mathbf{k})$.

The well-known Ohno-Zagier relation ([8]) is a class of relations about the sums of multiple zeta values of fixed weight, depth and height. For integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$, we denote by $X_0(k, n, s)$ the sum of all multiple zeta values of weight k , depth n and height s . The Ohno-Zagier relation says that

$$X_0(k, n, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

More explicitly, Ohno and Zagier gave the generating function expression

$$\sum_{k \geq n+s, n \geq s \geq 1} X_0(k, n, s) u^{k-n-s} v^{n-s} t^{s-1} \\ = \frac{1}{uv-t} \left\{ 1 - \exp \left(\sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (u^n + v^n - \alpha^n - \beta^n) \right) \right\},$$

where α and β are determined by $\alpha + \beta = u + v$ and $\alpha\beta = t$. In [7], we showed that the Ohno-Zagier relation can be deduced from the regularized double shuffle relation. In [6], we generalized the concept height to i -height, studied sums of multiple zeta values of fixed weight, depth and general height, and expressed a kind of generating function of these sums in terms of generalized hypergeometric functions.

Similarly, we denote by $X_0^*(k, n, s)$ the sum of all multiple zeta-star values of weight k , depth n and height s for integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$. The authors of [1] considered a generating function $\Phi_0^*(u, v, t)$ of sums $X_0^*(k, n, s)$, where

$$\Phi_0^*(u, v, t) = \sum_{k \geq n+s, n \geq s \geq 1} X_0^*(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.$$

It was proved in [1] that $\Phi_0^*(u, v, t)$ can be expressed by a special value of the generalized hypergeometric function ${}_3F_2$ as

$$(1.1) \quad \Phi_0^*(u, v, t) = \frac{1}{(1-v)(1-\beta)} {}_3F_2 \left(\begin{matrix} 1-\beta, 1-\beta+u, 1 \\ 2-v, 2-\beta \end{matrix}; 1 \right),$$

where α, β are determined by $\alpha + \beta = u + v, \alpha\beta = uv - t^2$, and the generalized hypergeometric function ${}_3F_2$ is defined as (see [3])

$${}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n (\alpha_3)_n}{n! (\beta_1)_n (\beta_2)_n} z^n,$$

with the Pochhammer symbol $(a)_n$ given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0, \\ a(a+1) \cdots (a+n-1), & \text{if } n > 0. \end{cases}$$

Similarly to [6], the authors of [2] considered a kind of generating function of sums of multiple zeta-star values of fixed weight, depth and general height, and represented this generating function via generalized hypergeometric functions.

Since the generating function $\Phi_0^*(u, v, t)$ is represented by ${}_3F_2$ as in (1.1), it is expected that in general $X_0^*(k, n, s)$ can't be written as a polynomial of zeta values with rational coefficients. While in [5] Kaneko and Ohno considered some kind of duality of multiple zeta-star values, and proposed the following conjecture.

Kaneko-Ohno Conjecture ([5]). *For any positive integers m, n, s with $m, n \geq s$, we have*

$$(-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), \dots].$$

It was proved in [5] that the conjecture is true for $s = 1$. Using the result of [1] about the generating function $\Phi_0^*(u, v, 0)$, Yamazaki gave another proof of this case

in [9]. Note that the Kaneko-Ohno theorem for their conjecture in the case $s = 1$ can be restated as

$$(1.2) \quad u\Phi_0^*(-u, v, 0) - v\Phi_0^*(-v, u, 0) \\ = \frac{1}{u} - \frac{1}{v} + \frac{\Gamma(u+v)}{\Gamma(u)\Gamma(v)}((\Gamma(v)\Gamma(1-v))^2 - (\Gamma(u)\Gamma(1-u))^2).$$

The purpose of this paper is to give a proof of Kaneko-Ohno Conjecture. In fact, similarly to (1.2), we give an expression of $u\Phi_0^*(-u, v, t) - v\Phi_0^*(-v, u, t)$ by gamma functions in Theorem 2.2. Our proof is based on the expression of $\Phi_0^*(u, v, t)$ given in [1], and hence is similar to the one of Yamazaki given in [9] for the special case $s = 1$.

In Section 2, we state our main result and give some corollaries. In Section 3, we prepare a result about generalized hypergeometric series ${}_3F_2$. In the last section, we give the proof of the main theorem.

2. STATEMENT OF THE MAIN RESULT

2.1. Main theorem. As in Section 1, we denote by $X_0^*(k, n, s)$ the sum of all multiple zeta-star values of weight k , depth n and height s for integers k, n, s with $k \geq n + s$ and $n \geq s \geq 1$. Let $\Phi_0^*(u, v, t)$ be the generating function defined by

$$\Phi_0^*(u, v, t) = \sum_{k \geq n+s, n \geq s \geq 1} X_0^*(k, n, s) u^{k-n-s} v^{n-s} t^{2s-2}.$$

For variables u, v, t , we define a and b by the conditions $a + b = -u + v$ and $ab = -uv - t^2$. Equivalently, we have

$$a, b = \frac{-u + v \pm \sqrt{(u+v)^2 + 4t^2}}{2}.$$

After that we define the function $A(u, v, a, b)$ by

$$(2.1) \quad A(u, v, a, b) = \frac{1}{2\pi} \left\{ \frac{\cos \pi u}{\sin \pi v} - \frac{\cos \pi v}{\sin \pi u} + \cos \pi(a-b)(\cot \pi u - \cot \pi v) \right\}.$$

Note that $A(u, v, a, b) = A(u, v, b, a)$, which shall play an important role in the proof of our main theorem. We can express $A(u, v, a, b)$ by gamma functions as in the following lemma.

Lemma 2.1. *We have*

$$(2.2) \quad A(u, v, a, b) = \frac{1}{\Gamma(u+a)\Gamma(1-u-a)} \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(a)\Gamma(1-a)} + \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(b)\Gamma(1-b)} \right),$$

and

$$(2.3) \quad A(u, v, a, b) = \frac{1}{\Gamma(u+b)\Gamma(1-u-b)} \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(b)\Gamma(1-b)} + \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(a)\Gamma(1-a)} \right).$$

Proof. Equation (2.3) follows from equation (2.2) and the fact $A(u, v, a, b) = A(u, v, b, a)$. Using the well-known reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

we find that the right-hand side of equation (2.2) becomes

$$\frac{\sin \pi(u+a)}{\pi} \left(\frac{\sin \pi a}{\sin \pi v} + \frac{\sin \pi b}{\sin \pi u} \right),$$

which is equal to

$$\frac{1}{2\pi} \left(\frac{\cos \pi u - \cos \pi(v + a - b)}{\sin \pi v} + \frac{\cos \pi(u + a - b) - \cos \pi v}{\sin \pi u} \right).$$

Now it is easy to finish the proof. \square

The main theorem of this paper is the following theorem.

Theorem 2.2. *We have*

$$(2.4) \quad u\Phi_0^*(-u, v, t) - v\Phi_0^*(-v, u, t) \\ = \frac{u-v}{ab} + A(u, v, a, b) \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}.$$

2.2. Some remarks. By the definition of the generating function $\Phi_0^*(u, v, t)$, it is easy to see that

$$(2.5) \quad u\Phi_0^*(-u, v, t) - v\Phi_0^*(-v, u, t) \\ = \sum_{\substack{m, n \geq s \\ s \geq 1}} (-1)^s ((-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s)) \\ \times u^{m+1-s} v^{n+1-s} t^{2s-2} + \sum_{n \geq s \geq 1} (-1)^{n+s} X_0^*(n+s, s, s) (u^{n+1-s} - v^{n+1-s}) t^{2s-2}.$$

Since we have the expansion

$$\Gamma(1-x) = \exp \left(\gamma x + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} x^n \right),$$

where γ is the Euler's constant, we know that Theorem 2.2 indeed implies Kaneko-Ohno Conjecture.

Corollary 2.3. *For any positive integers m, n, s with $m, n \geq s$, the difference*

$$(-1)^m X_0^*(m+n+1, n+1, s) - (-1)^n X_0^*(m+n+1, m+1, s)$$

can be expressed as a polynomial of zeta values with rational coefficients.

Pay attention to the second term of the right-hand side of equation (2.5), we have another corollary.

Corollary 2.4. *For any positive integers k, s with $k \geq 2s$, the sum $X_0^*(k, s, s)$ can be expressed as a polynomial of zeta values with rational coefficients.*

Note that the above corollary is an immediate consequence of the symmetric sum formula for multiple zeta-star values (see [4, Theorem 2.1]).

Let $t = 0$ in Theorem 2.2, we can get equation (1.2). In fact, in this case, we can assume that $a = -u$ and $b = v$. For $A(u, v, a, b)$, we use the equivalent equation (2.2). Then using Theorem 2.2, we really get equation (1.2).

3. A RESULT ABOUT GENERALIZED HYPERGEOMETRIC SERIES ${}_3F_2$

To prove the main theorem of this paper, we introduce the following result.

Proposition 3.1. *Let $a, b, c \in \mathbb{C}$ with their real parts sufficient small. We have*

$$(3.1) \quad {}_3F_2 \left(\begin{matrix} a, b, c \\ a+b, 1+c \end{matrix}; 1 \right) \\ = \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} (\psi(1+c-b) - \psi(a) - \psi(b) - \gamma) \\ - \frac{\Gamma(a+b)\Gamma(1+c)\Gamma(1+c-a-b)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a)\Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a)_n(1-b)_n}{nn!(1+c-b)_n},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

To save space, from now on we will denote the special value ${}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix}; 1 \right)$ by ${}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right)$.

To prove the above proposition, we need two transformation formulas. The first one is (see [3, Sec. 3.8, Eq. (1), p. 21])

$$(3.2) \quad {}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right) = \frac{\Gamma(\beta_1)\Gamma(\beta_1 - \alpha_1 - \alpha_2)}{\Gamma(\beta_1 - \alpha_1)\Gamma(\beta_1 - \alpha_2)} {}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \beta_2 - \alpha_3 \\ \alpha_1 + \alpha_2 - \beta_1 + 1, \beta_2 \end{matrix} \right) \\ + \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1 + \alpha_2 - \beta_1)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_2 - \alpha_3)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} \\ \times {}_3F_2 \left(\begin{matrix} \beta_1 - \alpha_1, \beta_1 - \alpha_2, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3 \\ \beta_1 - \alpha_1 - \alpha_2 + 1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 \end{matrix} \right),$$

provided that $\Re(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\Re(\alpha_3 - \beta_1 + 1) > 0$. The second one is (see [3, Ex. 7, p. 98])

$$(3.3) \quad {}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \right) = \frac{\Gamma(\beta_2)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3)}{\Gamma(\beta_2 - \alpha_3)\Gamma(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)} {}_3F_2 \left(\begin{matrix} \beta_1 - \alpha_1, \beta_1 - \alpha_2, \alpha_3 \\ \beta_1, \beta_1 + \beta_2 - \alpha_1 - \alpha_2 \end{matrix} \right),$$

provided that $\Re(\beta_1 + \beta_2 - \alpha_1 - \alpha_2 - \alpha_3) > 0$ and $\Re(\beta_2 - \alpha_3) > 0$.

Proof of Proposition 3.1. Taking a parameter ε , such that $|\varepsilon|$ is sufficient small, we have

$${}_3F_2 \left(\begin{matrix} a, b, c \\ a+b, 1+c \end{matrix} \right) = \lim_{\varepsilon \rightarrow 0} {}_3F_2 \left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix} \right).$$

Now we consider the series ${}_3F_2 \left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix} \right)$. Applying (3.2), we get

$$(3.4) \quad {}_3F_2 \left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix} \right) = \frac{\Gamma(a+b+\varepsilon)\Gamma(\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)} {}_3F_2 \left(\begin{matrix} a, b, 1-\varepsilon \\ 1-\varepsilon, 1+c-\varepsilon \end{matrix} \right) \\ + \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1-\varepsilon)\Gamma(1+c)} {}_3F_2 \left(\begin{matrix} a+\varepsilon, b+\varepsilon, 1 \\ 1+\varepsilon, 1+c \end{matrix} \right).$$

To the first ${}_3F_2$ -series in the right-hand side of (3.4), we apply the Gaussian summation formula (see [3, Sec. 1.3, Eq. (1)])

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n}{n!(\beta)_n} = \frac{\Gamma(\beta)\Gamma(\beta - \alpha_1 - \alpha_2)}{\Gamma(\beta - \alpha_1)\Gamma(\beta - \alpha_2)}$$

for $\Re(\beta - \alpha_1 - \alpha_2) > 0$, and apply (3.3) to the second ${}_3F_2$ -series in the right-hand side of (3.4), we obtain

$${}_3F_2 \left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix} \right) = \frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)} \\ - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\varepsilon\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} {}_3F_2 \left(\begin{matrix} \varepsilon, 1-b, a+\varepsilon \\ 1+\varepsilon, 1+c-b \end{matrix} \right).$$

To the ${}_3F_2$ -series in the right-hand side of the above equation, we split it into two terms as $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$. Then we see that ${}_3F_2 \left(\begin{matrix} a, b, c \\ a+b+\varepsilon, 1+c-\varepsilon \end{matrix} \right)$ is equal to

$$\frac{1}{\varepsilon} \left(\frac{\Gamma(1+\varepsilon)\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a+\varepsilon)\Gamma(b+\varepsilon)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b-\varepsilon)} \right. \\ \left. - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} \right) \\ - \frac{\Gamma(a+b+\varepsilon)\Gamma(1+c-\varepsilon)\Gamma(1+c-a-b-\varepsilon)}{\Gamma(a)\Gamma(b)\Gamma(1+c-a-\varepsilon)\Gamma(1+c-b)} \sum_{n=1}^{\infty} \frac{(a+\varepsilon)_n(1-b)_n}{(n+\varepsilon)n!(1+c-b)_n}.$$

Finally, let ε go to 0 to finish the proof. For the first two lines of the above expression, we use L'Hôpital's rule and the fact that $\psi(1) = -\gamma$. \square

4. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 2.2.

Using the result of Aoki-Kombu-Ohno ([1]) for the generating function $\Phi_0^*(u, v, t)$, we have the following lemma.

Lemma 4.1. *Let α and β be determined by $\alpha + \beta = u + v$ and $\alpha\beta = uv - t^2$. We have*

$$\Phi_0^*(u, v, t) = \frac{\Gamma(\beta - \alpha)\Gamma(1 - \beta)\Gamma(v)\Gamma(1 - v)}{\Gamma(1 - \alpha)\Gamma(1 + u - \alpha)\Gamma(1 + \alpha - u)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(1 - \beta)_n}{n!(1 + \alpha - \beta)_n} \frac{\alpha - u}{n + \alpha - u} \\ + \frac{\Gamma(\alpha - \beta)\Gamma(1 - \alpha)\Gamma(v)\Gamma(1 - v)}{\Gamma(1 - \beta)\Gamma(1 + u - \beta)\Gamma(1 + \beta - u)} \sum_{n=0}^{\infty} \frac{(\beta)_n(1 - \alpha)_n}{n!(1 + \beta - \alpha)_n} \frac{\beta - u}{n + \beta - u}.$$

Proof. The result of Aoki-Kombu-Ohno in [1] gives that

$$\Phi_0^*(u, v, t) = \frac{\Gamma(\beta - \alpha)\Gamma(1 - v)}{\Gamma(1 - \alpha)\Gamma(1 + u - \alpha)} \int_0^1 s^{-\beta}(1 - s)^{v-1} {}_2F_1 \left(\begin{matrix} \alpha, \alpha - u \\ 1 + \alpha - \beta \end{matrix} ; s \right) ds \\ + \frac{\Gamma(\alpha - \beta)\Gamma(1 - v)}{\Gamma(1 - \beta)\Gamma(1 + u - \beta)} \int_0^1 s^{-\alpha}(1 - s)^{v-1} {}_2F_1 \left(\begin{matrix} \beta, \beta - u \\ 1 + \beta - \alpha \end{matrix} ; s \right) ds.$$

Here ${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; s \right)$ is the Gaussian hypergeometric function given by

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; s \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} s^n.$$

Hence we have

$$\begin{aligned}
& \int_0^1 s^{-\beta} (1-s)^{v-1} {}_2F_1 \left(\begin{matrix} \alpha, \alpha-u \\ 1+\alpha-\beta \end{matrix}; s \right) ds \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\alpha-u)_n}{n! (1+\alpha-\beta)_n} \int_0^1 s^{n-\beta} (1-s)^{v-1} ds \\
&= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\alpha-u)_n}{n! (1+\alpha-\beta)_n} \frac{\Gamma(1+n-\beta) \Gamma(v)}{\Gamma(1+n+v-\beta)}.
\end{aligned}$$

Now it is easy to finish the proof. \square

Recall that we have defined a and b by

$$a + b = -u + v, \quad ab = -uv - t^2.$$

Using the above lemma, we immediately get the following result.

Lemma 4.2. *We have*

$$u\Phi_0^*(-u, v, t) - v\Phi_0^*(-v, u, t) = F(u, v, a, b) + F(u, v, b, a),$$

where $F(u, v, a, b)$ is defined by

$$\begin{aligned}
& \frac{\Gamma(b-a)}{\Gamma(1-u-a)\Gamma(1+u+a)} \left(\frac{u\Gamma(v)\Gamma(1-v)\Gamma(1-b)}{\Gamma(1-a)} \sum_{n=0}^{\infty} \frac{(a)_n (1-b)_n}{n! (1+a-b)_n} \frac{u+a}{n+u+a} \right. \\
& \left. - \frac{v\Gamma(u)\Gamma(1-u)\Gamma(1+a)}{\Gamma(1+b)} \sum_{n=0}^{\infty} \frac{(1+a)_n (-b)_n}{n! (1+a-b)_n} \frac{u+a}{n+u+a} \right).
\end{aligned}$$

Since we have

$$\sum_{n=0}^{\infty} \frac{(a)_n (1-b)_n}{n! (1+a-b)_n} \frac{u+a}{n+u+a} = \sum_{n=0}^{\infty} \frac{(a)_n (-b)_n}{n! (a-b)_n} \frac{(a-b)(u+a)(n-b)}{-b(n+a-b)(n+u+a)},$$

and

$$\frac{n-b}{(n+a-b)(n+u+a)} = \frac{-a}{u+b} \frac{1}{n+a-b} + \frac{v}{u+b} \frac{1}{n+u+a},$$

we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(a)_n (1-b)_n}{n! (1+a-b)_n} \frac{u+a}{n+u+a} \\
&= \frac{a(u+a)\Gamma(1+a-b)}{b(u+b)\Gamma(1+a)\Gamma(1-b)} - \frac{v(a-b)}{b(u+b)} {}_3F_2 \left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix} \right).
\end{aligned}$$

In the above, we have used Gaussian summation formula for Gaussian hypergeometric function at unit argument. Similarly, we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(1+a)_n (-b)_n}{n! (1+a-b)_n} \frac{u+a}{n+u+a} \\
&= \frac{b(u+a)\Gamma(1+a-b)}{a(u+b)\Gamma(1+a)\Gamma(1-b)} + \frac{u(a-b)}{a(u+b)} {}_3F_2 \left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix} \right).
\end{aligned}$$

Hence we get the following lemma.

Lemma 4.3. *We have*

$$F(u, v, a, b) = F_1(u, v, a, b) + F_2(u, v, a, b),$$

where $F_1(u, v, a, b)$ is defined by

$$F_1(u, v, a, b) = \frac{(u+a)\Gamma(b-a)\Gamma(1+a-b)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)} \left(\frac{u\Gamma(v)\Gamma(1-v)}{b\Gamma(a)\Gamma(1-a)} - \frac{v\Gamma(u)\Gamma(1-u)}{a\Gamma(b)\Gamma(1-b)} \right),$$

and $F_2(u, v, a, b)$ is defined by

$$F_2(u, v, a, b) = \frac{uv(a-b)\Gamma(b-a)}{(u+b)\Gamma(1-u-a)\Gamma(1+u+a)} \left(\frac{\Gamma(v)\Gamma(1-v)\Gamma(-b)}{\Gamma(1-a)} - \frac{\Gamma(u)\Gamma(1-u)\Gamma(a)}{\Gamma(1+b)} \right) {}_3F_2 \left(\begin{matrix} a, -b, u+a \\ a-b, 1+u+a \end{matrix} \right).$$

Now we begin to compute $F_1(u, v, a, b) + F_1(u, v, b, a)$ and $F_2(u, v, a, b) + F_2(u, v, b, a)$. For $F_1(u, v, a, b) + F_1(u, v, b, a)$, we have the following result.

Lemma 4.4. *The sum $F_1(u, v, a, b) + F_1(u, v, b, a)$ equals*

$$\frac{u-v}{ab} + \frac{(a-b)uv}{ab(u+a)(u+b)} \Gamma(b-a)\Gamma(1+a-b) A(u, v, a, b).$$

Proof. Using the reflection formula for gamma function, we see that $F_1(u, v, a, b) + F_1(u, v, b, a)$ is equal to

$$\Gamma(b-a)\Gamma(1+a-b) \left\{ \frac{\sin \pi(u+a)}{\pi(u+b)} \left(\frac{u \sin \pi a}{b \sin \pi v} - \frac{v \sin \pi b}{a \sin \pi u} \right) - \frac{\sin \pi(u+b)}{\pi(u+a)} \left(\frac{u \sin \pi b}{a \sin \pi v} - \frac{v \sin \pi a}{b \sin \pi u} \right) \right\}.$$

The term in the brace of the above expression is

$$(4.1) \quad \begin{aligned} & \frac{1}{2\pi(u+b)} \left(\frac{u(\cos \pi u - \cos \pi v \cos \pi(a-b) + \sin \pi v \sin \pi(a-b))}{b \sin \pi v} \right. \\ & \quad \left. - \frac{v(\cos \pi u \cos \pi(a-b) - \sin \pi u \sin \pi(a-b) - \cos \pi v)}{a \sin \pi u} \right) \\ & - \frac{1}{2\pi(u+a)} \left(\frac{u(\cos \pi u - \cos \pi v \cos \pi(b-a) + \sin \pi v \sin \pi(b-a))}{a \sin \pi v} \right. \\ & \quad \left. - \frac{v(\cos \pi u \cos \pi(b-a) - \sin \pi u \sin \pi(b-a) - \cos \pi v)}{b \sin \pi u} \right). \end{aligned}$$

Picking up the common factors, and noting the identities

$$\begin{aligned} \frac{1}{b(u+b)} - \frac{1}{a(u+a)} &= \frac{v(a-b)}{ab(u+a)(u+b)}, \\ \frac{1}{a(u+b)} - \frac{1}{b(u+a)} &= \frac{u(b-a)}{ab(u+a)(u+b)}, \\ \frac{u}{b(u+b)} + \frac{v}{a(u+b)} + \frac{u}{a(u+a)} + \frac{v}{b(u+a)} &= \frac{2(v-u)}{ab}, \end{aligned}$$

we see that the expression (4.1) becomes

$$\frac{uv(a-b)}{ab(u+a)(u+b)}A(u, v, a, b) + \frac{(v-u)\sin\pi(a-b)}{ab\pi},$$

which finishes the proof. \square

For $F_2(u, v, a, b) + F_2(u, v, b, a)$, we apply Proposition 3.1 to get the following result.

Lemma 4.5. *The sum $F_2(u, v, a, b) + F_2(u, v, b, a)$ equals*

$$\begin{aligned} & \frac{(b-a)uv}{ab(u+a)(u+b)}\Gamma(b-a)\Gamma(1+a-b)A(u, v, a, b) \\ & + A(u, v, a, b)\frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)}. \end{aligned}$$

Proof. Applying Proposition 3.1 to the ${}_3F_2$ -series in $F_2(u, v, a, b)$, we find that $F_2(u, v, a, b)$ becomes

$$\begin{aligned} & \frac{(a-b)\Gamma(a-b)\Gamma(b-a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)\Gamma(1-u-a)} \left(\frac{\Gamma(v)\Gamma(1-v)}{\Gamma(a)\Gamma(1-a)} - \frac{\Gamma(u)\Gamma(1-u)}{\Gamma(-b)\Gamma(1+b)} \right) \\ & \times \left(\psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right), \end{aligned}$$

which is just

$$\begin{aligned} & A(u, v, a, b)\frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \\ & \times \left(\psi(1+v) - \psi(a) - \psi(-b) - \gamma - \sum_{n=1}^{\infty} \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right). \end{aligned}$$

Hence using the fact $A(u, v, a, b) = A(u, v, b, a)$, we find $F_2(u, v, a, b) + F_2(u, v, b, a)$ becomes

$$\begin{aligned} & A(u, v, a, b)\frac{\Gamma(b-a)\Gamma(1+a-b)\Gamma(u+a)\Gamma(u+b)}{\Gamma(u)\Gamma(v)} \\ & \times \left\{ \sum_{n=1}^{\infty} \left(\frac{(1+a)_n(b)_n}{nn!(1+v)_n} - \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right) + \psi(b) - \psi(-b) + \psi(-a) - \psi(a) \right\}. \end{aligned}$$

It is easy to see that

$$\sum_{n=1}^{\infty} \left(\frac{(1+a)_n(b)_n}{nn!(1+v)_n} - \frac{(a)_n(1+b)_n}{nn!(1+v)_n} \right) = \frac{b-a}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{n!(1+v)_n},$$

which equals

$$\frac{b-a}{ab} \frac{\Gamma(1+u)\Gamma(1+v)}{\Gamma(1+u+a)\Gamma(1+u+b)} + \frac{1}{b} - \frac{1}{a}$$

by Gaussian summation formula. Applying the formulas

$$\psi(-x) - \psi(x) - \frac{1}{x} = \pi \cot \pi x,$$

and

$$\pi \cot \pi a - \pi \cot \pi b = \frac{\Gamma(a)\Gamma(1-a)\Gamma(b)\Gamma(1-b)}{\Gamma(b-a)\Gamma(1+a-b)},$$

we finish the proof. \square

Proof of Theorem 2.2. Theorem 2.2 follows from Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.5. \square

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